

Topics in quantum chaos of generic systems

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Abstract. We review the main ideas and results in the stationary problems of quantum chaos in generic (mixed) systems, whose classical dynamics has regular (invariant tori) and chaotic regions coexisting in the phase space. First we discuss the universality classes of spectral fluctuations (GOE/GUE for ergodic systems, and Poissonian for integrable systems). We explain the problems in the calculation of the invariant (Liouville) measure of classically chaotic components, which has recently been studied by Robnik et al (1997) and by Prosen and Robnik (1998). Then we describe the Berry-Robnik (1984) picture, which is claimed to become exact in the strict semiclassical limit $\hbar \rightarrow 0$. However, at not sufficiently small values of \hbar we see a crossover regime due to the localization properties of stationary quantum states where Brody-like behaviour with the fractional power law level repulsion is observed in the corresponding quantal energy spectra.

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1 Introduction

The main problem of stationary quantum chaos is the following: *Given a quantum Hamiltonian (operator) \hat{H} with infinitely many bound states, being the quantized object based on its classical analog $H = H(\mathbf{q}, \mathbf{p})$, the Hamiltonian H as a function of the N coordinates \mathbf{q} and momenta \mathbf{p} , what are the geometrical and statistical properties of the eigenfunctions, their Wigner functions, of the energy spectra and of the matrix elements of other quantal observables.*

Of course, we have the (stationary) Schrödinger equation, which sometimes we can even solve analytically, e.g. in cases of analytically solvable one-dimensional potentials or in cases of separable N -dimensional potentials (Landau and Lifshitz 1997), and sometimes we can solve the underlying eigenvalue problem numerically, at least in principle. However, the analytically solvable problems are very untypical, although quite important, because we can use them to explore their neighbourhood (in the functional space of Hamiltonians H) by means of a large variety of perturbational techniques, and such a neighbourhood includes classically nonintegrable systems which are typical (generic). This is in analogy of the KAM systems in classical mechanics (Gutzwiller 1990). The numerical techniques can be applied to almost all systems, but it turns out that as soon as the system is not (classically) integrable and solvable, also the numerical techniques must be quite sophisticated, especially if we ask for high lying eigenstates. To know the Schrödinger equation and to have the potentiality of solving it helps as little as the analogous situation in the classical dynamics where the potentiality of solving the Hamilton-Jacobi equation does not help very much in studying the global, qualitative and quantitative, properties of motion in generic nonintegrable Hamiltonian systems. This has been realized for the first time by Henri Poincaré, who has shown (see Poincaré 1993, Goroff 1993) that the gravitational three-body problem is indeed *nonintegrable* and this broken integrability can no longer warrant the existence of invariant tori everywhere in phase space, thereby giving the way to true chaotic motion, which cannot be embedded into smooth N -dimensional invariant surfaces. The emergence of classically chaotic motion gives rise to the notion of qualitative dynamics, which is apt to embrace the richness and variety of chaotic behaviour. To achieve that new methods are needed, both analytically and numerically. Concepts of Surface-of-Section (SOS) and similar ones become indispensable to work in the new science of nonlinear dynamics, dealing with the dynamical systems described by the set of M ordinary first order differential equations in their phase space. For Hamiltonian systems we have $M = 2N$ Hamilton equations. On the quantum side we face a precisely analogous problem: Given the (stationary) Schrödinger equation of a N -dimensional quantum system, whose classical analogue is not only nonseparable but also nonintegrable and thus

chaotic, we can hardly see the structure of the solutions (eigenstates, described by the wave function and the corresponding Wigner function) and their global properties. New approach is necessary, including the numerical one, to see and classify all the possible types of behaviour. On the theoretical analytical side the semiclassical methods are quite essential, and this line of thoughts goes back to the pioneering and classical work of Gutzwiller (Gutzwiller 1990 and references therein) and Percival (1973), further developed by many workers in classical and quantum chaos (Chirikov 1979, Casati and Chirikov 1994, Berry 1983, Giannoni *et al* 1991, Haake 1991, Bohigas and Giannoni 1984, Bohigas 1991). For a recent excellent review, covering not only quantum chaos, but also all related theoretical and experimental branches of physics see the paper by Weidenmüller and coworkers (Guhr *et al* 1998).

The purpose of this paper is to review the main methods and results in the field of *quantum chaos*, i.e. the study of the solutions of the Schrödinger equation connected with the classically nonintegrable and chaotic systems.

2 The main assertion of stationary quantum chaos

The main assertion of stationary quantum chaos is the following answer to the *main problem of quantum chaos* in the semiclassical limit of sufficiently small \hbar :

2.1 Classical integrability

The case **(I)** of **classically integrable quantal systems** \hat{H} :

If H is classically integrable, then the wave function is locally a superposition of a finite number of plane waves, the number of directions of the wave vector being equal to the number of possible momentum vectors \mathbf{p} through the coordinate point \mathbf{q} . Globally, the probability density is equal to the classical probability density obtained by projecting the N -dimensional invariant torus onto the configuration space \mathbf{q} , up to within the resolution scale of the order of one de Broglie wave length. The corresponding Wigner function² $W(\mathbf{q}, \mathbf{p})$ of the eigenstate is a delta function on the invariant torus labeled by the quantized classical action variable $\mathbf{I}(\mathbf{q}, \mathbf{p}) = \mathbf{I}_{\mathbf{n}}$, where \mathbf{n} is the quantum number multi-index $\mathbf{n} = (n_1, n_2, \dots, n_N)$ denoting the Maslov (EBK) quantized invariant N -torus, so that

²For the definition and properties of Wigner functions see section 3.

$$W(\mathbf{q}, \mathbf{p}) = \frac{1}{(2\pi)^N} \delta_N(\mathbf{I}(\mathbf{q}, \mathbf{p}) - \mathbf{I}_n) \quad (1)$$

where δ_f is the f -dimensional Dirac delta function, and in our case $f = N$. The eigenvalues of \hat{H} , i.e. the eigenenergies, in a small interval, after unfolding, that is after reducing the mean energy level spacing to unity, obey (typically) the Poissonian statistics (Robnik and Veble 1998): The probability $E(k, L)$ of observing k levels inside an interval of length L is given by³

$$E_{integrable} = E_{Poissonian}(k, L) = \frac{L^k}{k!} \exp(-L) \quad (2)$$

The untypical cases have measure zero, and are characterized by some number theoretic special properties like e.g. the rectangle billiards with rational squared sides ratio.

2.2 Classical ergodicity

The case (E) of **classically ergodic quantal systems** \hat{H} :

If H is classically ergodic system, then the wave function is locally a superposition of infinitely many plane waves, the directions of the wave vector \mathbf{k} being isotropically distributed on a N -dimensional sphere of radius $\hbar^{-1} \sqrt{2m(E - V(\mathbf{q}))}$, if the Hamiltonian is $H(\mathbf{q}, \mathbf{p}) = \mathbf{p}^2/(2m) + V(\mathbf{q})$, where m is the mass and V the potential energy. Due to the ergodicity the phases of the plane waves are assumed to be random (Berry 1977a,b), which implies that the wave amplitude $\psi(\mathbf{q})$ is a Gaussian random function. The random phase assumption, however, can break down in vicinity of isolated unstable classical periodic orbits or families of such orbits, where we observe *the scars* (Heller 1984), i.e. the regions of enhanced probability density $|\psi|^2$. Thus, to the leading approximation we have the microcanonical Wigner function of *almost all* eigenstates (Shnirelman 1979, Berry 1977a,b, Voros 1979),

$$W(\mathbf{q}, \mathbf{p}) = \frac{\delta_1(E - H(\mathbf{q}, \mathbf{p}))}{\int d^N \mathbf{q} d^N \mathbf{p} \delta_1(E - H(\mathbf{q}, \mathbf{p}))} \quad (3)$$

³It can be shown that knowledge of all $E(k, L)$ -statistics is equivalent to the complete knowledge of all n -point correlation functions. The calculation of $E(k, L)$, however, is much easier than the calculation of correlation functions etc., since there is no binning and other advantages. This has been pointed out very clearly by Aurich *et al* (1997). See also the book by Mehta (1991).

The eigenenergies of \hat{H} in a small interval, after unfolding, obey the predictions of classical *Random Matrix Theories* (RMT), namely the statistics of the eigenvalues of the ensemble of orthogonal Gaussian matrices (GOE) or of unitary Gaussian ensembles (GUE) (depending on the existence or nonexistence of an antiunitary symmetry), introduced by Wigner, and also of the COE/CUE of Dyson (see e.g. the book by Haake (1991)). (We consider only systems with classical analog H and therefore ignore spin and GSE). This assertion has been proposed by Bohigas, Giannoni and Schmit (1984). It implies that $E(k, L)$ statistics must obey the RMT laws, the so-called BGS-Conjecture

$$E_{ergodic}(k, L) = E_{RMT}(k, L) \quad (4)$$

2.3 Classically mixed systems

The case **(M)** of **classically mixed (generic) quantal systems** \hat{H} :

If H is classically mixed system, then we can distinguish between regular and irregular states. Percival (1973) was the first to propose such a qualitative characterization of eigenstates. The regular states are associated with classical invariant tori (semi-classically EBK/Maslov quantized tori, to the leading semiclassical approximation), and the chaotic states are associated with chaotic components. This view has been made more quantitative in the work of Berry and Robnik (1984). The Berry-Robnik picture rests upon the *The Principle of Uniform Semiclassical Condensation* (PUSC, see section 3), which states that the Wigner functions of quantal states in the limit $\hbar \rightarrow 0$ become positive definite, and since they are mutually orthogonal, they must "live" on disjoint supports, and the phase space volume (Liouville measure) of each of them is of the order of $(2\pi\hbar)^N$. See section 3 and e.g. (Robnik 1997). The question is, what is the geometry of the object on which they "condense", and the answer - as a conjecture - is: uniformly on a classical invariant object (Berry 1977a,b, Robnik 1988, 1995, 1997). Therefore we have regular and irregular states. The assumption is that there is no correlation between the spectral sequences (regular and a series of irregular states). If $N \geq 3$ we have only one chaotic component (the Arnold web of chaotic motion pervades the entire phase space - energy surface - and is dense, i.e. its closure is the energy surface) and one associated irregular sequence of eigenstates, whereas in $N = 2$ we have many, even infinite number of sequences of irregular states, of smaller and smaller invariant measure, each sequence being associated with one chaotic component. It is thus assumed that the Wigner function of a regular state is of type (1), whilst for irregular states, and generally, it is

$$W(\mathbf{q}, \mathbf{p}) = \frac{\delta_f(\mathbf{F}(\mathbf{q}, \mathbf{p}))\chi_\omega(\mathbf{q}, \mathbf{p})}{\int d^N \mathbf{q} d^N \mathbf{p} \delta_f(\mathbf{F}(\mathbf{q}, \mathbf{p}))\chi_\omega(\mathbf{q}, \mathbf{p})} \quad (5)$$

where $\chi_\omega(\mathbf{q}, \mathbf{p})$ is the characteristic function of the invariant component, labeled by ω , being a (either smooth or nonsmooth, generally possibly also fractal) subset of the smooth $(2N - f)$ -dimensional invariant surface defined by the f implicit equations (global integrals of motion), namely $\mathbf{F}(\mathbf{q}, \mathbf{p}) = 0$, where $\mathbf{F} = (F_1, F_2, \dots, F_f)$. The characteristic function $\chi_\omega(\mathbf{q}, \mathbf{p})$ is defined to have value unity on ω and zero elsewhere. The integer number f can be anything between 1 (ergodic system) and N (integrable system).

Obviously, the formula (5) is the most general expression for a condensed Wigner function of a (pure) eigenstate. It generalizes the cases (I) and (E). Namely, if we have ergodicity, then $f = 1$, we put $F_1(\mathbf{q}, \mathbf{p}) = E - H(\mathbf{q}, \mathbf{p})$, and $\omega =$ entire energy surface, and we recover equation (3). In the other extreme (I), we have N global integrals of motion in involution, and so $\mathbf{F}(\mathbf{q}, \mathbf{p}) = \mathbf{I}_n - \mathbf{I}(\mathbf{q}, \mathbf{p})$, and $\omega =$ the invariant torus labeled by \mathbf{I}_n , and we recover equation (1) of case (I). In the most general case, therefore, formula (5) applies. Obviously, W is normalized (see next section),

$$\int d^N \mathbf{q} d^N \mathbf{p} W(\mathbf{q}, \mathbf{p}) = 1 \quad (6)$$

For generic (mixed) systems the most typical case is $f = 1$, $F_1(\mathbf{q}, \mathbf{p}) = E - H(\mathbf{q}, \mathbf{p})$ and ω is a (nonsmooth, typically fractal, chaotic) subset of the energy surface F_1 . We write down this most important case explicitly:

$$W(\mathbf{q}, \mathbf{p}) = \frac{\delta_1(E - H(\mathbf{q}, \mathbf{p}))\chi_\omega(\mathbf{q}, \mathbf{p})}{\int d^N \mathbf{q} d^N \mathbf{p} \delta_1(E - H(\mathbf{q}, \mathbf{p}))\chi_\omega(\mathbf{q}, \mathbf{p})} \quad (7)$$

It is important to know the relative invariant (Liouville) measure of chaotic and regular eigenstates because the Hilbert space of a mixed Hamiltonian system is split into regular and irregular eigenstates, in the strict semiclassical limit, precisely in proportion to the classical invariant measure of the integrable component (invariant tori) and of the irregular components.

It is quite obvious by looking at the equation (7) that the invariant Liouville measure of a subset ω of the energy surface is equal to

$$\rho(\omega) = \frac{\int d^N \mathbf{q} d^N \mathbf{p} \delta_1(E - H(\mathbf{q}, \mathbf{p}))\chi_\omega(\mathbf{q}, \mathbf{p})}{\int d^N \mathbf{q} d^N \mathbf{p} \delta_1(E - H(\mathbf{q}, \mathbf{p}))} \quad (8)$$

The relative invariant Liouville measure of the regular components will be denoted by ρ_1 , and the measures of chaotic components (ordered in sequence of decreasing measure) by $\rho_2, \rho_3, \dots, \rho_m$, where $m = \infty$ for $N = 2$ and $m = 2$ for $N \geq 3$, as already explained. In section 4 I shall explain how one can calculate the measures ρ_2, ρ_3, \dots

Assuming the above mentioned absence of correlations pairwise between m spectral sequences, due to the fact that they have disjoint supports and thus do not interact, where m is infinite for $N = 2$ and 2 for $N \geq 3$, the spectral statistics can be written as

$$E_{mixed}(k, L) = \sum_{k_1+k_2+\dots+k_m=k} \prod_{j=1}^m E_j(k_j, \rho_j L) \quad (9)$$

which is a manifestation of Berry-Robnik (1984) picture. Here $E_j(k, L)$ is $E_{Poisson}(k, L)$ for $j = 1$, and $E_{RMT}(k, L)$ for $j = 2, 3, \dots, m$. See cases **(I)** and **(E)**, equations (2) and (4). The picture is based on the reasonable assumption that (after unfolding) the mean density of levels in the j -th sequence of levels is ρ_j , simply applying the Thomas-Fermi rule of filling the phase space volume with elementary cells of size $(2\pi\hbar)^N$ in the thin energy shell embedding the corresponding subset ω . Therefore, please note that the second argument of $E_j(k, L)$ is weighted precisely by the classical relative invariant measure of the underlying invariant component. Also, if there were several regular (Poissonian) sequences they can be lumped together into a single Poissonian sequence (which we traditionally label by 1 with relative invariant measure ρ_1): It is easy to show, that if $\alpha_1, \alpha_2, \dots, \alpha_l$ are positive real numbers and β being their sum, $\beta = \alpha_1 + \alpha_2 + \dots + \alpha_l$, then for all k , and L ,

$$E_{Poisson}(k, \beta L) = \sum_{k_1+k_2+\dots+k_l=k} E_{Poisson}(k_1, \alpha_1 L) E_{Poisson}(k_2, \alpha_2 L) \dots E_{Poisson}(k_l, \alpha_l L) \quad (10)$$

by simply using the definition of $E_{Poisson}(k, L)$ of equation (2). Thus, we have some kind of a central limit theorem, saying that the statistically independent superposition of Poisson sequences results in a Poisson sequence, such that the total density of Poissonian levels β is equal to the sum of the partial level densities α_j , $j = 1, 2, \dots, l$.

The case **(M)** is the most general one, and as the limiting extreme cases includes cases **(I)** and **(E)**.

2.4 Limitations of the universality

There are two important limitations of the above stated asymptotic behaviour as $\hbar \rightarrow 0$, when \hbar is not yet small enough: One is the existence of the outer energy scale, and the other one is the localization phenomena. As for the first, it has been shown by Berry (1985), applying the semiclassical Gutzwiller periodic orbit theory (Gutzwiller 1990 and the references therein), that at energy scales (after unfolding!) $L \geq L_{max}$ we do not have the universality but typically a saturation, i.e. $E(k, L)$ statistics at L larger than

$$L_{max} = \frac{\hbar}{T_0 \langle \Delta E \rangle} \quad (11)$$

where $\langle \Delta E \rangle$ is the mean energy level spacing, and T_0 the period of the shortest classical periodic orbit in the dynamical system $H(\mathbf{q}, \mathbf{p})$, deviate from their universal behaviour of cases **(I)** and **(E)**, equations (2) and (4). Instead, e.g. the sigma and delta statistics become constant. (For definition and inter-relationship see section 3.) However, please observe that as \hbar goes to zero, also L_{max} goes to infinity as a power \hbar^{-N+1} , so that in the semiclassical limit the universality region becomes larger $\propto \hbar^{-N+1}$.

As for the second limitation bordering the universal behaviour we comment the following. If the value of the (effective) Planck constant \hbar is not sufficiently small, then the eigenfunctions might not be fully extended in the sense of the corresponding Wigner functions obeying the equations (1), (3) and (5), but can be localized (not uniformly extended) on the classical invariant object on which they condense. Such a deviation from the ultimate limiting semiclassical behaviour is therefore a manifestation of the localization phenomena in stationary eigenstates of autonomous Hamiltonian systems, and is manifested also in the spectral statistics. For example, if we have a classically ergodic system, but with very slow chaos (very large diffusion time), we shall observe strongly localized states, mimicking a regular integrable system. In such an extreme case of localization we shall observe Poissonian spectral statistics rather than GOE/GUE. Depending on the strength of localization we shall therefore be able to see transition from Poissonian to GOE/GUE behaviour in an ergodic system. In the intermediate crossover regime we observe Brody-like behavior with fractional power law level repulsion. In a KAM regime the same is true if the effective \hbar is not small enough to resolve the structure of small chaotic components. We shall describe these phenomena in the subsequent sections.

However, just briefly, a qualitative comment is in order at this place. The relevant criterion for localization is, that the so called *break time* or *Heisenberg time*,

defined through

$$t_{break} = t_H = \frac{2\pi\hbar}{\langle\Delta E\rangle} \quad (12)$$

where $\langle\Delta E\rangle$ is the mean level spacing, must be shorter than the diffusion time, so then we have strongly localized states, whilst in the opposite extreme we have strongly extended states. The reason is very simple: Quite generally quantum mechanics (of a suitably chosen initial wave packet) follows classical dynamics (of a suitably chosen ensemble of initial conditions) up to the break time, after which the interference phenomena set in, resulting typically in destructive interference, and thus in the stop of diffusion, which means localization (before the entire phase space has been conquered). For example, in two-dimensional billiards, $\langle\Delta E\rangle$ is just constant, so the break time is constant and independent of energy, whilst the classical diffusion time, even if very large at small energies E , decreases with energy as $const./\sqrt{E}$, so ultimately, as $E \rightarrow \infty$, we shall find the extended states and then the general picture of case **(M)** is applicable, which, of course, as the extreme cases, includes **(I)** and **(E)**. However, the phenomena of localization, including the scars, are extremely important and as we have sketched above, they are related to the important time scales which control the finite time structure and behaviour of classical dynamics, especially the transport times and so on.

2.5 Distribution and fluctuation properties of transition probabilities

Finally, in this subsection we should comment on the statistical properties of the matrix elements of other observables in the eigenbasis of an integrable, ergodic and mixed system. The main work in this direction has been done by Feingold and Peres (1986) for the ergodic case, and this has been generalized to integrable and mixed systems by Prosen and Robnik (1993a).

The expectation values and generally the matrix elements of other reasonable observables (Hermitian operators having a classical limit) have been little studied (Feingold and Peres 1986, Alhassid and Levine 1986, Wilkinson 1987, 1988). One well known result concerns the fluctuation properties of generalized intensities (squares of matrix elements) within the framework of random matrix theories, namely the Porter-Thomas distribution (Brody *et al* 1981), which has been experimentally observed and suggested by Porter and Thomas (1956) in the context of nuclear physics. We expect that this fluctuation law applies also in classically ergodic systems with

few freedoms. The main motivation of our work (Prosen and Robnik 1993a) was to explain this and to find the appropriate generalization for Hamiltonian systems in the transition region of mixed dynamics.

In order to study the fluctuation properties of generalized intensities one must be able to clearly separate the smooth mean part of the intensities as the function of frequency (= energy difference between the final and initial state/ \hbar) from its fluctuating part. So, given the frequency of the intensity we ask what is its mean value and which is the distribution of its fluctuating part in units of the mean value. In the classically ergodic case Feingold and Peres (1986) propose a formula expressing the mean intensities in terms of the power spectrum of the given observable taken over a dense chaotic classical orbit. In deriving this result they rely on the Shnirelman theorem (1979) expressing the quantum expectation value of a reasonable operator as the classical microcanonical average. This theorem is obvious once one has in mind that the Wigner distributions of the eigenstates of a classically ergodic system in the semiclassical limit are just microcanonical distributions (Berry 1977a,b, Voros 1979), equation (3). In order to rederive Feingold-Peres formula and to generalize it we first point out that the Shnirelman theorem applies also to the states in the regular and mixed regime if the classical average is taken over the relevant classical invariant ergodic component, which supports the corresponding semiclassical eigenstate. This can be an invariant torus, a chaotic component, or the entire energy surface.

Following Feingold and Peres (1986) we start by looking at the following sum over eigenstates k of eigenenergies E_k for the transition elements $A_{jk} = \langle j | \hat{A} | k \rangle$

$$\begin{aligned} \sum_k \exp(i(E_j - E_k)t/\hbar) |A_{jk}|^2 &= \sum_k \langle j | e^{iE_j t/\hbar} \hat{A} | k \rangle \langle k | e^{-iE_k t/\hbar} \hat{A} | j \rangle = \\ &= \langle j | e^{i\hat{H}t/\hbar} \hat{A} e^{-i\hat{H}t/\hbar} \hat{A} | j \rangle = \langle j | \hat{A}(t) \hat{A}(0) | j \rangle \end{aligned} \quad (13)$$

Now we apply the generalized Shnirelman theorem, stating that in the *semiclassical limit* this is equal to the classical average

$$C_j(t) = \{A(t)A(0)\}_j \quad (14)$$

over the invariant ergodic component labeled by j which supports the semiclassical state $|j\rangle$. Using the ergodicity on the given invariant component this two-point autocorrelation function can be expressed as the time average along a classical dense orbit (dense in the given invariant component, which e.g. can be an invariant torus, or a chaotic component, or the entire energy surface)

$$C_j(t) = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} d\tau A(t + \tau) A(\tau). \quad (15)$$

Next we replace the sum \sum_k by the integral $\int dE_k \rho(E_k)$, where $\rho(E)$ is the density of states, and perform the Fourier transform and obtain

$$\langle |A_{jk}|^2 \rangle_j = \frac{S_j((E_k - E_j)/\hbar)}{2\pi\hbar\rho(E_k)} \quad (16)$$

where the state j is fixed and the average $\langle \cdot \rangle_j$ is taken over states k within a thin energy shell of thickness of few mean level spacings. Here

$$S_j(\omega) = \int_{-\infty}^{\infty} dt C(t) e^{i\omega t} = \lim_{T \rightarrow \infty} \frac{1}{T} \left| \int_{-T/2}^{T/2} dt A(t) e^{i\omega t} \right|^2 \quad (17)$$

is the power spectrum of a dense orbit in the invariant ergodic component j . If A has a nonvanishing mean value $\{A\}_j$ the $S_j(\omega)$ will have a delta spike at $\omega = 0$, and this can be removed by replacing A in the above formulas by $A - \{A\}_j$. To calculate the actual mean values of the intensities $|A_{jk}|^2$ we perform in the above formula (on the LHS) also the averaging over the j states microcanonically over the thin energy shell around E_j of sufficiently wide width such that the corresponding semiclassical states uniformly cover the energy surface, whilst on the RHS we correspondingly take the microcanonical average over all initial conditions j on the energy surface E_j . So the final formula for the mean generalized intensities is

$$\langle |A_{jk}|^2 \rangle = \frac{\{S((E_j - E_k)/\hbar)\}_E}{2\pi\hbar\rho(E)} \quad (18)$$

By $\{\cdot\}_E$ we denote the microcanonical average over the energy surface E . The apparent asymmetry in jk of this formula disappears in the semiclassical limit $\hbar \rightarrow 0$. In the numerical evaluations described in (Prosen and Robnik 1993a) we applied the above formula with $\{S(\omega)\}_E$ and $\rho(E)$ being calculated on the energy surface placed half way between E_j and E_k , i.e. $E = (E_j + E_k)/2$. This choice is met to minimize the error at finite \hbar .

Knowing the average value of intensities as a function of ω we can now separate the smooth part from its fluctuating part by renormalizing the matrix elements as follows

$$X_{jk} = \frac{A_{jk}}{\sqrt{\langle |A_{jk}|^2 \rangle}}. \quad (19)$$

The renormalized matrix elements X_{jk} are now regarded as random variable whose probability distribution is denoted by $D(X)$, which by definition has unit dispersion, and naturally is expected to be even function of X , $D(X) = D(-X)$, and so it has zero mean. In the classically ergodic case we expect that quite generally the matrix elements of a given operator are very well modelled by the GOE of

random matrix theories (Brody *et al* 1981) which predict Gaussian distribution for $D_{PT}(X) = \exp(-X^2/2)/\sqrt{2\pi}$, which is equivalent to the so-called Porter-Thomas distribution for the intensities $I = X^2$, namely $P(I) = \exp(-I/2)/\sqrt{2\pi I}$, see (Porter and Thomas 1956). In integrable cases one expects a vast abundance of at least approximate selection rules which render most X to become zero implying that $D(X)$ approaches a delta function $\delta(X)$ in the semiclassical limit. This can be seen by considering matrix representation of an operator in the basis of the torus quantized eigenstates of an integrable system, as explained in detail in (Prosen and Robnik 1993b). In the mixed type dynamics (KAM) in the transition region between integrability and chaos we expect a continuous transition from $\delta(X)$ towards $D_{PT}(X)$. More precisely, a semiclassical formula for $D(X)$ in such transition region has been derived by Prosen (1994a) by taking into account the fact that the only broadening of $D(X)$ stems from the transitions between chaotic initial and chaotic final states belonging to the same family of the invariant ergodic components (continuously parametrized by the energy), while all other transitions are almost forbidden. This work rests upon a more detailed analysis of higher autocorrelation functions and is reported on in (Prosen 1994b). More details and the numerical illustration of our results can be found in (Prosen and Robnik 1993a) and in (Prosen 1994a,b).

3 The Principle of Uniform Semiclassical Condensation and more about the wave functions and statistics

In this section we want to explain the main ingredients and arguments leading to the equations (1), (3) and (5). The ideas go back to Berry (1977a,b), Shnirelman (1979), Voros (1979), Robnik (1988, 1995, 1997). To see that the quantum analogy of the stationary (aspects of) chaos works well it is necessary to look at the objects uniquely determined by given eigenstates in such a manner that one can compare the eigenstates to the classical states (phase portraits at given energy). This can be achieved by introducing the Wigner functions (transforms) of given eigenstates, e.g. of the wave functions.

With this procedure we are building up a kind of the quantal phase space, in the spirit of the Wigner-Weyl formalism (de Groot and Suttrop 1972), in the following way: Let $\psi_n(\mathbf{q})$ be the n -th wave function (eigenfunction as a solution of the Schrödinger problem) in the N -dimensional configuration space with \mathbf{q} being a position vector, then its corresponding Wigner function (or transform) is defined as:

$$W_n(\mathbf{q}, \mathbf{p}) = \frac{1}{(2\pi\hbar)^N} \int d^N \mathbf{X} \exp(-\frac{i}{\hbar} \mathbf{p} \cdot \mathbf{X}) \psi_n(\mathbf{q} - \frac{\mathbf{X}}{2}) \psi_n^*(\mathbf{q} + \frac{\mathbf{X}}{2}) \quad (20)$$

where $*$ denotes the complex conjugation. The Wigner function is obviously real. However, unlike the classical distribution functions, the Wigner functions are *not* positive definite, which is a fundamental consequence of the very nature of quantum mechanics: If they were positive then the quantum mechanics would be identical to classical mechanics. This is the essence of the Wigner theorem (about the quantal phase space distributions). On the other hand, the Wigner functions do have the correct property that they become the configurational probability density $|\psi(\mathbf{q})|^2$ when projected down onto the configuration space (i.e. integrating (20) over the momenta \mathbf{p}) and complementary, they become the momentum probability density $|\varphi(\mathbf{p})|^2$ when integrated over the entire configuration space, as can be immediately verified from the definition (20). Therefore W_n integrates to 1 if ψ_n is normalized to unity, i.e.

$$\int W_n(\mathbf{q}, \mathbf{p}) d^N \mathbf{q} d^N \mathbf{p} = 1. \quad (21)$$

They also obey the orthogonality relation

$$(2\pi\hbar)^N \int d^N \mathbf{q} d^N \mathbf{p} W_n(\mathbf{q}, \mathbf{p}) W_m(\mathbf{q}, \mathbf{p}) = \delta_{nm}, \quad (22)$$

(δ_{nm} here is the Kronecker delta, i.e. discrete delta function, equal to 1 if $n = m$ and zero otherwise) which can be understood at once by recalling that $(2\pi\hbar)^N W_n$ is in fact the Weyl symbol of the projection operator $P_n = |n\rangle\langle n|$. Further, it can be easily seen that the absolute value of W_n is bounded from above, namely (Baker 1958, see also de Groot and Suttrop 1972, and Berry 1977b)

$$|W_n(\mathbf{q}, \mathbf{p})| \leq \left(\frac{1}{\pi\hbar}\right)^N \quad (23)$$

showing that it can diverge only in the semiclassical limit when $\hbar \rightarrow 0$. One can also see from (22) that (when $n = m$) one has

$$\int W_n^2(\mathbf{q}, \mathbf{p}) d^N \mathbf{q} d^N \mathbf{p} = 1/(2\pi\hbar)^N \quad (24)$$

Therefore unlike (21) the latter integral (24) can diverge as $\hbar \rightarrow 0$, and this divergence can be due to large contributions for large values of $|\mathbf{q}|$ and $|\mathbf{p}|$ or due to the singularities of W_n^2 (Berry 1977a). The latter possibility is the one that actually occurs, as shown by Baker (1958)

$$W_n(\mathbf{q}, \mathbf{p}) \rightarrow (2\pi\hbar)^N W_n^2(\mathbf{q}, \mathbf{p}), \quad \text{as } \hbar \rightarrow 0. \quad (25)$$

Therefore in the semiclassical limit the Wigner functions W_n become positive definite and divergent $\approx (2\pi\hbar)^{-N}$, which is weak enough still to obey (22) with $n = m$. This orthogonality relation shows then that all pairs W_n, W_m , with $n \neq m$ must have disjoint supports. Therefore they are effectively nonzero and divergent only on a small piece of volume of size $(2\pi\hbar)^N$. In fact this semiclassical condensation must take place close to and on the energy shell around the classical energy surface of energy E_n , of total volume $(2\pi\hbar)^N$. This is of course equivalent to the simple Thomas-Fermi rule of how to determine the average density of states, semiclassically: Divide the available classical phase space volume inside the given energy E_n by $(2\pi\hbar)^N$, and on the average this must be equal to n , which is the sequential quantum number, i.e. the (cumulative) number of eigenstates below the energy E_n . From these general considerations we cannot conclude more than stated. It is not clear a priori on what geometrical object does the Wigner function W_n condense as $\hbar \rightarrow 0$. Berry (1977a,b) and Voros (1979), in agreement with Shnirelman's theorem (1979), have suggested that in case of classical ergodicity W_n condenses *uniformly* on the energy surface, becoming the microcanonical distribution, (3). On the other hand, in the opposite extreme of an integrable motion, or a KAM system with invariant tori, which are EBK quantized and support the quantum state $|n\rangle$, Berry (1977a) derives from the semiclassical wave functions in coordinate space that the corresponding Wigner function is equal to (1). So here the Wigner function condenses uniformly on the EBK quantized torus.

The latter result can be easily obtained by noting that in the semiclassical limit the classical canonical transformations and the quantization do commute, and thus can be performed directly in the space of action-angle variables, immediately yielding the above result, as shown in (Robnik 1995, Hasegawa *et al* 1989).

In both extreme cases, the ergodicity (3), and the quasi-integrability (existence of a quantized KAM torus) (1), we see that the Wigner function condenses uniformly on the underlying classical object, which is the invariant indecomposable component in the classical phase space. It seems thus very natural to elevate these findings to the *Principle of Uniform Semiclassical Condensation*, PUSC, (Robnik 1988, Li and Robnik 1994, Robnik 1997), which claims the following: *In the semiclassical limit $\hbar \rightarrow 0$ the Wigner function of the n -th eigenstate condenses uniformly on the underlying classical invariant object (topologically transitive component) labeled by ω , which can be an invariant N -torus, the entire energy surface (in case of ergodicity), or a chaotic component. The corresponding Wigner function is, in the most general case, given by (5).* This principle has a great predictive power, when accepted.

The statistical properties of the wave functions in the coordinate space have been analyzed by Berry (1977b), where he has shown that in ergodic cases the proba-

bility amplitude distribution is Gaussian random functions, and its autocorrelation function is described by the Bessel functions. In case of integrability or quasi-integrability (a quantized invariant KAM torus) the wave functions in coordinate space are quite ordered, they typically have caustics (projection singularities when projecting the Wigner function (1) down the momentum space onto the configuration space). They are locally well described by the *finite* number of plane waves, because classically there is only a finite number of possible trajectory velocities (obtained by projection of the torus and of the quasiperiodic orbits on the torus onto the configuration space), whereas in case of ergodicity the number of plane waves is *infinite* (a circular ensemble of wave vectors), and they have uncorrelated phases, which implies Gaussian randomness. See e.g. (Robnik 1988, 1995, 1997). The limiting behaviour of the condensing Wigner function on a classical invariant object (no \hbar enters in this equation, so in a sense we have classical Wigner functions!) implies also that the coarse grained probability density in the configuration space is just classical probability density, obtained by projecting the Wigner function onto the coordinate space, i.e. by integrating it over the momenta \mathbf{p} . By coarse grained we mean smoothing over a few de Broglie wavelengths. And by the classical probability density we mean the value proportional to the time spent (asymptotically) in each cell of equal size (relative invariant measure) in the discretized phase space. Recently, we have brilliantly demonstrated this fact (Robnik *et al* 1998).

From the behaviour of the stationary wave functions we now turn back to the study of the statistical properties of the energy spectra, based on PUSC, giving some more details than in section 2. Again we restrict our discussion only to the time independent (autonomous) Hamiltonian systems with finite (bounded) classical motion and correspondingly a purely discrete quantal energy spectrum (no scattering states). Further we assume that there are infinitely many energy levels, so that the questions of statistical properties of energy spectra can be raised. Van Kampen (1985) has defined quantum chaos as "*...that property that causes a quantum system to behave statistically*". Now we have seen that this element is involved in the morphology of the eigenstates of classically chaotic, especially ergodic, systems. Therefore we must conclude that *stationary quantum chaos* exists, and corresponds exactly to the classical chaos. We shall now demonstrate that this is the case also when studying the energy spectra (and possibly also the statistical properties of other observables).

One of the most important cornerstones of the stationary quantum chaos is the so-called *Bohigas-Giannoni-Schmit Conjecture* (1984), BGS-Conjecture, introduced already in subsection 2.2, equation (4). It states that the classically ergodic Hamiltonian systems (with discrete spectrum) exhibit universal spectral fluctuations, whose statistical properties are correctly captured by the conventional Random Matrix

Theories (RMT) (Mehta 1991), and are thus universal. See case (E) of section 2. If we ignore spin (which is not important in studying the classical limit) then the spectral fluctuations are described by either the fluctuations of the eigenvalues in the Gaussian Orthogonal Ensemble (GOE) of random matrices, if the system has an antiunitary symmetry, or by Gaussian Unitary Ensemble (GUE), if there is no antiunitary symmetry, such as e.g. time reversal symmetry, involved in the system. Of course, this statement applies to the statistical analysis of the spectrum, after the unfolding procedure, in which the actual physical energy is replaced by the average number of states up to the given energy - i.e. the integrated (cumulative) level distribution. After unfolding the mean energy level spacing is by construction equal to unity. The average density of states typically is very well described by the familiar Thomas-Fermi rule of filling the classical phase space volume with the quantum cells of size $(2\pi\hbar)^N$, for which we have seen the reason in the above analysis of the semiclassical behaviour of the condensed Wigner functions of eigenstates. Sometimes the corrections to this *asymptotically exact* rule can be obtained, e.g. in plane and N -dimensional billiards, constituting the famous Weyl rule with perimeter, curvature, corner corrections etc (See e.g. Berry and Howls 1994 and references therein). In this unfolding procedure the information on the (nonuniversal) average density of states is eliminated from the spectrum, giving way to the possibility of *universal fluctuations of energy levels around its nonuniversal mean distribution*. In the classically ergodic systems this is exactly what we find, confirmed and supported by many numerical and actual experiments, and theoretically first corroborated by the result of Berry (1985) on the delta (Δ) statistics, and recently claimed to be proven by Andreev *et al* (1996), however, under much stronger conditions than ergodicity, namely assuming the exponential decay of correlations. For some recent review see (Robnik 1994,1995,1997). An important recent work in this connection is by Keating *et al* (1996,1997).

What we said in the above paragraph applies to the scaling systems, where the energy limit (of quantum number $n \rightarrow \infty$) is somehow equivalent to the semiclassical limit of $\hbar \rightarrow 0$. That is, there is a scaling variable involving the energy and \hbar such that the classical dynamics is constant while energy is changing. One such example are the billiard systems, among which the plane billiards are most widely used models. Examples of rigorously ergodic systems are the Sinai billiard, the stadium billiard of Bunimovich and the cardioid billiard. The latter is the limiting case of the family of billiards with analytic boundaries defined as the quadratic conformal map of the unit disc, introduced by Robnik (1983,1984) and further studied by many workers. See Robnik *et al* (1997). In billiards the topology and the geometry of the phase portrait are exactly identical at all energies except for the scaling of the momentum as a square root of the energy. Therefore the limit of infinite energy is equivalent to the semiclassical limit of $\hbar \rightarrow 0$. The constancy of the classical

dynamics across the spectral stretches that we study is important to draw clear and safe conclusions about the relationship between the classical and quantum chaos. If a system is not a scaling system, then there is no way out other than taking just a small energy interval and letting $\hbar \rightarrow 0$ so that in this limit the interval is containing an arbitrarily large number of energy levels, a necessary condition to introduce and to define statistical distributions. This general case is the one that we assumed in section 2.

There are two most important statistical measures used to characterize the energy spectra, and both of them are easily related to the $E(k, L)$ statistics that we introduced in section 2. One is the *level spacing distribution* usually denoted $P(S)$, where S is the length of the spacing and $P(S)dS$ is the probability that S lies within the infinitesimal interval $(S, S + dS)$. It is normalized to unit probability (by definition of probability density) and to unit first moment (due to the construction by the unfolding procedure). It can be shown that $P(S)$ is the second derivative of the so-called gap probability $E(0, L)$ of having no levels inside the interval of length $S = L$, so $P(S) = d^2 E(0, S)/dS^2$. $P(S)$ measures the short range correlations between energy levels. Here the important point is the behaviour of $P(S)$ at small S : For GOE one has the linear behaviour $P(S) \approx \text{const} \times S$ whilst for GUE we have $P(S) \approx \text{const} \times S^2$. Correspondingly we talk about the *linear* and *quadratic level repulsion*: Because $P(S) \rightarrow 0$ as $S \rightarrow 0$ the level crossings, i.e. the degeneracies, are not likely, and in GUE this level repulsion is stronger than in GOE, giving rise - paradoxically - to a more regular spectrum. For GOE/GUE the corresponding $P(S)$ for the infinite dimensional case cannot be obtained in a closed form, and for the details the reader is referred e.g. to (Bohigas 1991). However, it is quite surprising and fortunate that the 2-dim GOE/GUE models yield closed analytic formulae which give an excellent approximation to the infinite dimensional case (which normally is referred to when speaking about RMT results). For GOE we have the so-called Wigner distribution

$$P_{\text{GOE}}(S) = \frac{\pi S}{2} \exp\left(-\frac{\pi S^2}{4}\right), \quad (26)$$

and for the GUE case

$$P_{\text{GUE}}(S) = \frac{32S^2}{\pi^2} \exp\left(-\frac{4S^2}{\pi}\right). \quad (27)$$

Both cases are easily derived by assuming the 2-dim Gaussian real symmetric matrices (for GOE) or Hermitian symmetric matrices (for GUE). Interestingly, they can be derived from the so-called Wigner surmise (see e.g. Bohigas and Giannoni 1984, Brody *et al* 1981), which is an approximate argument outside the scope of RMT; it

is some kind of a statistical argument.

Long range correlations are measured by the second most important statistical measure, the delta statistics $\Delta(L)$, introduced by Dyson and Mehta. It is an inverse measure of spectral rigidity/regularity, as is immediately obvious from the definition:

$$\Delta(L) = \langle \min_{A,B} \frac{1}{L} \int_{-L/2}^{L/2} [\mathcal{N}(x) - Ax - B]^2 dx \rangle \quad (28)$$

where $\mathcal{N}(x)$ is the *unfolded* cumulative spectral staircase function ($\mathcal{N}(x)$ = linear average x plus oscillatory part $\tilde{\mathcal{N}}(x)$), the minimum is taken with respect to the parameters A and B , and the average denoted by $\langle \dots \rangle$ is taken over a suitable energy interval over x . Thus from this very definition $\Delta(L)$ is the local average least square deviation of the spectral staircase $\mathcal{N}(x)$ from the best fitting straight line over an energy range x of L mean level spacings. The more regular the (unfolded) spectrum, the easier is to find a linear fit over L levels, and consequently the smaller is $\Delta(L)$. The fact that we try to find the best linear fit to the spectral staircase implies that at small L the delta statistics $\Delta(L)$ always behaves as $L/15$ and thus carries no information about the system at all.

Sometimes it is useful also to know the number variance, denoted by $\Sigma^2(L)$, the dispersion of the number of levels $n(L)$ in an interval of length L , where $\langle n(L) \rangle = L$, and

$$\Sigma^2(L) = \langle (n(L) - L)^2 \rangle = L - 2 \int_0^L (L - r) Y_2(r) dr \quad (29)$$

where $Y_2(r)$ is the pair cluster function (Bohigas 1991, Haake 1991, Mehta 1991). There exists also the connection between Σ^2 and Δ ,

$$\Delta(L) = \frac{2}{L^4} \int_0^L (L^3 - 2L^2r + r^3) \Sigma^2(r) dr \quad (30)$$

although, strictly speaking, this has been proven so far only within the context of RMT (Aurich *et al* 1997, Mehta 1991). Finally, as announced in section 2, we shall consider not the set of all the cluster functions $Y_n(x_1, x_2, \dots, x_n)$, where $n = 2, 3, \dots$, but rather the $E(k, L)$ statistics, for all $k = 0, 1, 2, \dots$, following the suggestion of Steiner and coworkers (Aurich *et al* 1997), because they are very easy to calculate numerically and yet contain the complete information about the spectral statistics. Since by definition $E(k, L)$ is *the probability* that inside an interval of length L we find exactly k levels, there are simple relationships to other statistical quantities. For example, as already mentioned, the level spacing distribution $P(S)$ is

$$P(S) = \frac{\partial^2}{\partial L^2} E(k=0, L=S) \quad (31)$$

and

$$\Sigma^2(L) = \sum_{k=0}^{\infty} (k-L)^2 E(k, L). \quad (32)$$

and therefore, through (30), we have the relation expressing $\Delta(L)$ in terms of the $E(k, L)$ statistics.

From the definition of the Poissonian $E(k, L)$ statistics in (2) it is easily derived (see (31))

$$P_{Poisson}(S) = \exp(-S) \quad (33)$$

and after (32)

$$\Sigma_{Poisson}^2(L) = L, \quad (34)$$

and then using (30)

$$\Delta_{Poisson}(L) = \frac{L}{15}. \quad (35)$$

Poissonian statistics means also by definition that there are no correlations, i.e. the pair correlation function factorizes, so that we have for the pair cluster function (c.f. Mehta 1991, Bohigas 1991)

$$Y_2^{Poisson}(x) = 0. \quad (36)$$

Thus, using this fact in equation (29) and then (30) we again recover Poissonian values (34) and (35).

It is at large L that the different universality classes of behaviour emerge. Interesting are the asymptotic results for large L . For a completely regular (equidistant) spectrum (like e.g. one-dimensional harmonic oscillator) one obtains that for large $L \gg 1$, $\Delta(L)$ is just constant and equal to $1/12$. On the other hand the RMT gives for GOE

$$\Delta_{GOE}(L) \approx \frac{1}{\pi^2} \log L, \quad (37)$$

and for GUE

$$\Delta_{\text{GUE}}(L) \approx \frac{1}{2\pi^2} \log L, \quad (38)$$

This result of RMT has been derived on dynamical grounds for the energy spectra of *individual* classically ergodic systems, applying the Gutzwiller's periodic orbit theory, in a remarkable and important paper by Berry (1985), giving some theoretical support to BGS-Conjecture. As it is believed that the level repulsion is a purely quantal effect and cannot be derived semiclassically (Robnik 1986, 1989), one is not surprised that for eleven years there was hardly any theoretical progress in establishing the BGS-Conjecture. Indeed, Berry concluded (1991) that $P(S)$ cannot be derived from applying periodic orbit theory, because it depends sensitively on orbits of all lengths (periods). However, recently Andreev *et al* (1996) claim to have derived BGS-Conjecture by a different thinking, namely by studying the spectrum of the Frobenius-Peron operator in the semiclassical limit, using the techniques from the supersymmetry field theories, especially the nonlinear sigma model (Weidenmüller *et al* 1985). In considering the semiclassical limit they assume stronger properties than ergodicity, namely the exponential decay of (classical) correlations.

We may conclude that the support for the BGS-Conjecture is so strong that it can be regarded as well established, although not rigorously proven as yet. By this I mean especially the unusually strong and massive numerical support and evidence accumulated during the past fourteen years. Thus given the correctness of BGS-Conjecture we speak about *the universality classes of spectral fluctuations*, namely the GOE and GUE class, of subsection 2.2. As is seen in the above formulae for $P(S)$ and for $\Delta(L)$, and in the general equation (4), the universality is indeed established: There is no parameter in the statistical properties of spectral fluctuations, and the statistical measures are identical for all ergodic systems, irrespective of their dynamical and geometrical details. Turning this aspect around, we conclude that the spectral fluctuations in a classically ergodic system do not have any further information content. When talking about BGS-Conjecture as applied to individual classically ergodic systems we must emphasize that if there are any exact unitary symmetries involved in the system, then they must be first eliminated before applying BGS-statement. This process we call desymmetrization. Then, even after desymmetrization, we still have to decide whether GOE or GUE statistics apply: The general classification criterion is: If the system has an antiunitary symmetry (like e.g. the time reversal symmetry) then GOE applies, and GUE otherwise (i.e. if there is no antiunitary symmetry) (Robnik and Berry 1986, Robnik 1986).

It is a surprise that RMT apply so well to *individual* dynamical systems. We claimed

that BGS-Conjecture holds true in the strict semiclassical limit $\hbar \rightarrow 0$. However, how small must be \hbar to see this happening? Thus we are now addressing the question of the limitations to universality due to the not-sufficiently-small value of the (effective) Planck constant.

The following criterion is important: As soon as the eigenstates are fully extended chaotic (ergodic) in the sense of (3) the BGS-Conjecture applies. This is always happening asymptotically, as $\hbar \rightarrow 0$, but a rough criterion is that the classical diffusion time (the typical time for the classical dynamics to conquer the entire available phase space - energy surface) is shorter than the break time introduced in section 2.4, equation (12): If this inequality applies (strong enough, i.e. by a factor 10 or so) then the eigenstates will be fully extended. If the inequality is reversed, then the eigenstates are chaotic (they lie in a classically chaotic region) but *localized*, which means occupying only a small piece (a proper subset) of the dynamically available phase space. Obviously the break time $(2\pi\hbar)/\langle\Delta E\rangle$ goes to infinity as $\hbar \rightarrow 0$, whilst the classical transport time is independent of \hbar , and thus the desired inequality is asymptotically always satisfied, and therefore in the limit all semiclassical states are fully extended states and we recover the universality of section 2.

In such a dynamically localized classically ergodic regime another interesting phenomenon occurs, namely the fractional power law level repulsion, by which we mean that

$$P(S) \approx \text{const} \times S^\beta, \quad (39)$$

where the exponent β can be 0, 1, 2 or anything in between. Thus the localization phenomena of the chaotic eigenfunctions soften the strength of the level repulsion (β going from 1 to 0). This phenomenon seems quite obvious, since the tails of the localized wave functions overlap even less and thus interact less strongly resulting in reducing the value of β . This trend towards the Poisson statistics is well known in the context of localization phenomena in disordered solid state systems. At present we do not have a theory on how β should be related to the localization lengths/areas/volumes, and how to predict them. But we have numerical examples demonstrating these features (see e.g. Prosen and Robnik 1994a,b). The theory must satisfy the known limit of $\beta \rightarrow 1$ when $\hbar \rightarrow 0$. One distribution function which at present does not have any deep physical justification as yet, but is just a nice mathematical model which captures the global level spacing distribution with the local property (39) is the well known Brody distribution (Brody 1973, Brody *et al* 1981)

$$P_{\text{Brody}}(S; \beta) = aS^\beta \exp(-bS^{\beta+1}), \quad a = b(\beta + 1), \quad b = \{\Gamma(\frac{\beta + 2}{\beta + 1})\}^{\beta+1} \quad (40)$$

where a and b are obviously determined by the normalizations of the total probability and of the first moment to unity. For $\beta = 0$ we have Poisson distribution (exponential, i.e. $P(S) = \exp(-S)$), and for $\beta = 1$ we have Wigner (i.e. 2-dimensional GOE, given in equation (26)). The role of dynamical localization phenomena has been first realized by Chirikov *et al* (1981) in the time-dependent systems like kicked rotator and Rydberg atoms in microwaves, but has been also suggested in the time-independent Hamiltonian systems (Chirikov 1993). Feingold (1996) has recently found deeper relationship between the two phenomena.

After having explained the two universality classes of spectral fluctuations in the classically ergodic systems, we now have to add and rediscuss the third universal class comprising of classically integrable systems of two or more degrees of freedom (Robnik and Veble 1998), see subsection 2. (Systems with only one degree of freedom are exceptional and special in the sense that as $\hbar \rightarrow 0$ the local spectrum is just the perfectly regular equidistant spectrum.) Indeed, if we have two or more quantum numbers the entire spectrum can be thought of as being composed of an infinite number of statistically uncorrelated number sequences, which of course must result in the Poisson statistics $E(k, L)$ in (2), and specifically (33), (34) and (35).

There are semiclassical arguments resting upon the torus quantization by Berry and Tabor (1977) showing that the statistics should be Poissonian. The torus quantization (EBK quantization) is embodied in equation (1), describing the associated Wigner functions. However, these semiclassical arguments are only approximation and it is far from obvious that they should correctly describe e.g. the fine structure of energy spectra and thus the level repulsion and their absence, as has been recently pointed out by Prosen and Robnik (1993c,d). In fact, as explained above, it is believed that semiclassics cannot explain the level repulsion (short range correlations), since it is a purely quantum effect. We know that there are exceptions from the Poissonian behaviour which have been rigorously proven to exist and involve some highly nontrivial and sophisticated mathematics (Bleher *et al* 1993). Nevertheless, there is quite massive numerical and experimental support to the statement that the spectral fluctuations of classically integrable systems are quite accurately described by the Poisson statistics (2) (Robnik and Veble 1998). There might be cases where the statement is rigorous, whereas in general we think that the measure of exceptions is small and maybe vanishing in some sense. In every case this delicate problem persists to be very important and difficult, but it should also be accepted that typically Poisson model is an excellent approximation. The most important feature is the absence of short range correlations implying the absence of level repulsion,

which means that degeneracies are allowed and this is mathematically exhibited in $P(S) \rightarrow \text{const} \neq 0$, in fact according to (33) we have $P(S) \rightarrow 1$ as $S \rightarrow 0$.

Another remark should be made about a limitation to universality, mentioned in subsection 2.4, the behaviour of $\Delta(L)$ in individual dynamical systems at large L , where the limitations to universality of section 2.4 set in. As discovered by Casati *et al* (1985) and later explained by Berry (1985) there is the phenomenon of saturation, by which we mean that $\Delta(L)$ becomes effectively constant and equal Δ_∞ if $L > L_{\max}$, in *any* system (ergodic, integrable, partially chaotic - KAM type). This leveling off of the delta statistics is nonuniversal, but the L_{\max} can be estimated in the context of Berry's theory (1985) as in equation (11). Therefore for any $N \geq 2$ the onset of saturation L_{\max} goes to infinity in the semiclassical limit $\hbar \rightarrow 0$, giving way to the full universality of the three universality classes (37, 38, 35). The details of the saturation value Δ_∞ at a fixed and nonzero \hbar can be found in (Berry 1985).

Finally, we should say something more about the $P(S)$ and $\Delta(L)$ in the classically mixed systems, thus giving more details of the Berry-Robnik (1984) picture. Both statistics are of course implied by the most general statistics (9), through the formulae (31) for $P(S)$, and through (32) and (30) for sigma and delta statistics.

Using the general equation (9) and approximation (26) to first calculate E_{GOE} , and then to find $P(S)$ through (31), for m level sequences, we obtain an explicit analytic formula for $P(S)$, namely

$$P_m(S) = \frac{d^2}{dS^2} [\exp(-\rho_1 S) \prod_{j=2}^m \text{erfc}(\frac{\sqrt{\pi}}{2} \rho_j S)] \quad (41)$$

where $\text{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty dt \exp(-t^2)$ is the complementary error function. (For GUE one has to use (27) instead of (26).) One can show

$$P_m(S=0) = 1 - \sum_{j=2}^m \rho_j^2, \quad (42)$$

so that now as a consequence of the statistically independent superposition of a number of level (sub)sequences we get a trend towards Poissonian statistics, since the degeneracies become possible due to the lack of level repulsion among the regular levels on the one hand and among the levels belonging to different sequences. Thus, e.g. even if there is no regular component but two equally strong chaotic components, so that $\rho_1 = 0$, but $\rho_2 = \rho_3 = 1/2$, we get $P_m(S=0) = 1 - 1/4 = 3/4 \neq 0$. Only if there is only one chaotic (ergodic) component we find $P_m(S=0) = 0$, describing the GOE-like level repulsion.

Most important in practical applications is the 2-component Berry-Robnik formula ($m = 2$), namely

$$P_2(S, \rho_1) = \rho_1^2 \exp(-\rho_1 S) \operatorname{erfc}\left(\frac{\sqrt{\pi}}{2} \rho_2 S\right) + (2\rho_1 \rho_2 + \frac{1}{2} \pi \rho_2^3 S) \exp(-\rho_1 S - \frac{1}{4} \pi \rho_2^2 S^2), \quad (43)$$

and we see the special case of equation (42)

$$P_2(S = 0, \rho_1) = 1 - \rho_2^2 = \rho_1(2 - \rho_1), \quad (44)$$

which vanishes only iff $\rho_1 = 0$ and $\rho_2 = 1$ (ergodicity). This level spacing distribution is very important especially in practical applications, because in mixed systems typically we have a very large dominant chaotic region, so that the next largest chaotic region is much smaller by orders of magnitude, say only one percent of the leading one and can be neglected. In such case the two component formula (43) is an excellent approximation.

For the delta statistics (28) one can derive *the additivity property* implied by the statistical independence of the (sub)sequences. First one shows it for the sigma statistics (the number variance) (see e.g. Bohigas 1984,1991). Then one can show (Seligman and Verbaarschot 1985)

$$\Delta(L) = \sum_{j=1}^m \Delta_j(\rho_j L). \quad (45)$$

It is now interesting to verify whether Berry-Robnik theory applies to actual systems which we can analyze numerically. We (Prosen and Robnik 1993c,1994a,b) have done such analysis for a certain one-parameter family of billiards, introduced by Robnik (1983, 1984), namely the 2-dim billiard shape defined as the complex quadratic conformal mapping $w = w(z) = z + \lambda z^2$ of the unit disc $|z| \leq 1$ in the z -plane onto the w -plane. The boundary curve is known in the theory of analytic curves as the Pascal's Snail. (Usually workers refer to this billiard system as Robnik billiard, because it was introduced and dynamically analyzed by the author.) The system is very important because it has analytic boundaries up to the limiting value of the shape parameter $\lambda = 1/2$ where the singularity appears at $z = -1$, and therefore for small λ the KAM-Theory applies: At $\lambda = 0$ we have the integrable case of the circle billiard with conserved angular momentum. For small $\lambda \leq 1/4$ we

have a convex billiard with analytic boundaries, studied extensively especially by Lazutkin (1981,1991), where much can be said about the classical and semiclassical analysis, including a construction of an approximate integral of motion (Robnik and Berry 1985, Robnik 1986). This is essentially KAM scenario. For $\lambda > 1/4$ the boundary is nonconvex and the KAM theory does not apply because the bounce map becomes discontinuous. When $\lambda = 1/4$ the first point of zero curvature appears at $z = -1$ and according to Mather (1982) this guarantees that all the Lazutkin caustics (generated by invariant tori for glancing orbits supporting the whispering gallery modes of quantal eigenstates in the semiclassical picture) are destroyed, giving way (preparing the way) for ergodicity, which has been postulated by Robnik (1983). In fact, a careful analysis of certain periodic orbits (Hayli *et al* 1987) has shown that also for $\lambda > 1/4$ they can be stable, surrounded by very tiny stability islands that can hardly be detected numerically, which was the reason why in the early work (Robnik 1983) they have not been seen. According to their estimates the system has stable islands up to $\lambda \approx 0.2791$, whilst Li and Robnik (1996) have numerical evidence that ergodicity is possible for $\lambda \geq 0.2775$. Recently it has been rigorously proven by Markarian (1993) that for $\lambda = 1/2$ (we have a cusp singularity at $z = -1$, because $dw/dz = 0$ there) the system is ergodic, mixing and K. This is thus the first billiard system with analytic boundaries having the chance to be ergodic for $0.2775 < \lambda \leq 1/2$. (The Sinai billiard and the stadium of Bunimovich are rigorously ergodic, mixing and K, but they do not have analytic boundaries.) The system has been recently studied by many workers (Berry and Robnik 1986, Robnik and Berry 1986, Frisk 1990, Bruus and Stone 1994, Stone and Bruus 1993a,b, Bäcker *et al* 1995, Bruus and Whelan 1996).

The system is thus ideal to study the morphology of quantum eigenstates at various λ , following a continuous transition from the domain of torus states and Poisson statistics, through the regime of the generic behaviour of mixed dynamics, to the extreme case of (rigorous) ergodicity and entirely chaotic states with GOE statistics. (Of course, as explained in the introduction, we must separate the exact symmetry classes of a given dynamical system before performing the statistical analysis of the energy spectra. This procedure is called the desymmetrization. In our case we have even and odd reflection symmetry classes.) The main results on this have been published in (Li and Robnik 1995a,b,c)

For the details please see (Robnik 1997) and the references therein. Another brilliant numerical confirmation of the Berry-Robnik statistics was given in (Prosen and Robnik 1994a,b) for the quantized compactified standard map, and by Prosen (1995,1996) for a semiseparable oscillator, and recently for a quartic billiard (Prosen 1998). The main problems and issues concerning the Berry-Robnik picture have been expounded recently in our comment (Robnik and Prosen 1997). The deviation from

Berry-Robnik regime towards the Brody-like behaviour with fractional power law level repulsion, as described in subsection 2.4, has been analyzed in detail in (Prosen and Robnik 1993c, 1994a,b).

4 Statistical properties of classically chaotic motion and the measure of chaotic components

In this section we address the question of the statistics of classically chaotic motion and the problem of how to determine the measure of the chaotic components ρ_2, ρ_3, \dots , which in this section we shall simply denote by ρ_2 , relevant for the problem of stationary quantum chaos in mixed systems, dealt with in subsection 2.3.

In a recent work (Robnik *et al* 1997) we have demonstrated some general scaling laws in the behaviour of stochastic diffusion in strongly chaotic systems (ergodic, mixing and K with large Lyapunov coefficient, i.e. large KS entropy), mainly in Hamiltonian systems, or in the strange attractors of dissipative systems. The so-called *random model* that we developed describes very well the diffusion on chaotic components, in the sense that the relative (invariant) measure $\rho(j)$ as a function of the discrete time⁴ j approaches unity exponentially as

$$\rho(j) = 1 - \exp(-j/N_c) \quad (46)$$

where N_c is the number of cells of equal size (relative invariant measure) $a = 1/N_c$ into which the whole ergodic component is decomposed, provided N_c is sufficiently large, say $N_c > 100$ or so. In the above equation we have defined $\rho(j) = \rho_2(j)/\rho_2(\infty)$. Thus the average measure of occupied domain on the grid of cells is $\langle ka \rangle = \rho(j)$. This *random model* rests upon the assumption that there are absolutely no correlations, not even between two consecutive steps, so that at each step (of filling the N_c cells) we have the same *a priori probability* $a = 1/N_c$ of visiting any of the cells, irrespective of whether they are already occupied or not. Such absence of correlations can be implied and expected by the large Lyapunov exponents, which in turn imply strong stretching and folding (of a phase space element) even after one iteration, meaning that such a phase element will be evenly distributed (in the coarse grained sense) over the entire phase space (or surface of section). The universal scaling property is reflected in the fact that $\rho(j)$ is a function of the ratio (j/N_c) only, and does not depend on j and N_c separately.

⁴We work either with mappings or with Poincaré mappings on the surface of section. In each case j is the number of the iterations of the map.

Such an assumption of absence of all correlations appears to be strong at first sight, and therefore it is quite surprising that the model describes a whole lot of deterministic dynamical systems for which we can expect large Lyapunov exponents, namely 2D billiard (Robnik 1983, $\lambda = 0.375$), 3D billiard (Prosen 1997a,b, $a = -\frac{1}{5}$, $b = -\frac{12}{5}$), ergodic logistic map (tent map), hydrogen atom in a strong magnetic field ($\epsilon = -0.05$) (Robnik 1981, 1982, Hasegawa *et al* 1989), and standard map at ($k = 400$), in which the agreement is almost perfect, except for the last two systems where we see some long-time deviations on very small scales. However, in the standard map at $k = 3$, and in Hénon-Heiles (1964) system at $E = \frac{1}{6}$ the deviations are noticeable though not very big (about only 1%).

It is also quite astonishing that the random model applies very well even to *ergodic-only* systems, with strictly zero Lyapunov exponents, namely in case of the rectangle billiards (Artuso *et al* 1997), where the deviations from the exponential law (46) on the largest scale is within a few percent only. It is a well known result (Sinai 1976) that polygonal billiards have exactly zero Lyapunov exponents, easy to understand since all periodic orbits are marginally stable (parabolic), and since they are everywhere dense, we conclude that the Lyapunov exponents must be zero everywhere.

As a small but interesting comment we should mention our results on testing the random number generators from (Press *et al* 1986), where two of them (ran0 and ran3) are found to be in perfect agreement with the random model, whilst the other two (ran1 and ran2) are exhibiting big deviations. Thus, indeed, some deterministic dynamical systems like hydrogen atom in strong magnetic field etc. can be better number generators than some built-in (black-box) computer algorithms. It should be acknowledged, however, that there are other random number generators which pass all tests of randomness, including ours, e.g. in (Finocchiario *et al* 1993).

The random model developed in (Robnik *et al* 1997) is a statistical model which predicts not only the average relative measure of occupied cells $\rho(j) = \langle ka \rangle$, the average taken over k , in (46), but also the standard deviation $\sigma(j)$, which under the same assumption of sufficiently large N_c is equal to, to the leading order,

$$\sigma(j) = \sqrt{\langle (ka)^2 \rangle - \langle ka \rangle^2} = \sqrt{\frac{1 - \rho(j)}{N_c}}, \quad (47)$$

and gives us an estimate of the size of expected statistical fluctuations in $\rho(j)$.

The random model (Robnik *et al* 1997) has been subsequently generalized in an important direction (Prosen and Robnik 1998), namely to describe the diffusion on

chaotic components in systems of mixed dynamics, with divided phase space, having regular regions (invariant tori) coexisting in the phase space with chaotic regions, a typical KAM scenario (Kolmogorov 1954, Arnold 1963, Moser 1962, Benettin *et al* 1984, Gutzwiller 1990). Such systems in two degrees of freedom can have the fractal boundary between the regular and irregular component and thus the convergence to the theoretically expected results can be very slow, mimicking a departure from the random model, although ultimately it conforms to this model. In three or more degrees of freedom there is no boundary between the regular and chaotic regions, because we have the Arnold web (Chirikov 1979), which is everywhere dense in the phase space, and thus a naive box-counting would imply always that the relative invariant measure of the chaotic component is equal to the measure of the entire phase space, so $\rho_2(j) = 1$, which is wrong, because the KAM theorem gives rigorously that the relative measure of the regular component ρ_1 is strictly positive, $\rho_1 > 0$, moreover it is close to unity with the perturbation parameter. We assume that the invariant measure of the chaotic component is positive, although strictly speaking this is a major open theoretical problem in the mathematics of nonlinear systems, the so-called coexistence problem (Strelcyn 1991). Therefore in such case one must introduce the possibility of different *a priori probabilities*, which now are no longer just the same and equal to $a = 1/N_c$, but have a certain distribution described by the so-called greyness distribution $w(g)$, where g is a continuous variable on the interval $[0, 1]$: $g = 0$ means no visits (white cells), $g = 1$ are the most frequently occupied cells (black cells), and those cells with $0 < g < 1$ have intermediate number of visits (grey cells). With this model we have shown how by measuring (numerically calculating) $w(g)$ we can determine the relative invariant measure μ of the chaotic component. The result is

$$\mu = \int_0^1 gw(g)dg \quad (48)$$

and the time dependent relative measure of occupied domain is equal to

$$\rho(j) = 1 - \int_0^1 dgw(g) \exp(-\frac{gj}{\mu N_c}) \quad (49)$$

and the standard deviation is still given precisely by the equation (47). The greyness distribution can be calculated numerically quite easily by noticing that the greyness g is proportional to the average occupancy number $n(g)$, namely $n(g) = g/\mu$, so by measuring $n(g)$ in the limit $j \rightarrow \infty$ and after normalizing the g of the peak of $n(g)$ to unity, we get the g 's, and then by binning them into bins of suitably small size Δg we get the histogram for $w(g)$.

In case of ergodicity (only one chaotic component) we have $w(g) = \delta(g - 1)$, the Dirac delta function at $g = 1$, and then from equations (48) and (49) follows the random model, with exponential behavior (46).

In a later work we dealt with only *ergodic* systems, but such having several components, each of them also being ergodic, but weakly coupled, by which we mean that the transition probability for going from one to another component is very small and the typical transition time j^* very long. Obviously, at small times we shall find the random model (46) with N_c being equal to the number of cells of the starting component, $N_c = N_1$, whilst for very large times j , bigger than the typical transition time j^* , so $j \gg j^*$, we shall find again the random model (46), but now with N_c being equal to the number of all cells in the system, $N_c = N_s$. In between, when $j \approx j^*$, we have the crossover regime which we analyse in the present work. So, the finite time structure of ergodic systems controlled by their transport times can be also captured analytically. We call the corresponding model the multi-component random model of diffusion. For more details see (Robnik *et al* 1998).

As the concluding remark of this section it should be explained that the relative measures $\rho_1, \rho_2, \dots, \rho_m$ we need in the formula for the statistics of the classically mixed systems in subsection 2.3, equation (9), are the relative Liouville measures *on the energy surface* (E), the surface being defined by $H(\mathbf{q}, \mathbf{p}) = E = \text{const.}$, whereas in this section we explained how to calculate the relative (invariant symplectic) measure μ of a given invariant chaotic set ω *on the Surface of Section* (SOS). They are not the same, for a given chaotic set, but the procedures to calculate them are relatively simply related to each other, as has been explained by Meyer (1985). The answer is obtained from the general relation

$$\langle A \rangle_E = \int_E d^N \mathbf{q} d^N \mathbf{p} A(\mathbf{q}, \mathbf{p}) \delta_1(E - H(\mathbf{q}, \mathbf{p})) = \langle \tau A \rangle_{SOS} = \int_{SOS} dX \tau A(\mathbf{q}, \mathbf{p}), \quad (50)$$

where dX is the invariant symplectic measure on SOS, saying, that the microcanonical average of any classical function $A(\mathbf{q}, \mathbf{p})$ (observable) over the energy surface E is equal to the average over the SOS of A weighted by the average time of recurrence τ of a trajectory to SOS on each invariant (ergodic) component. τ is constant for a given trajectory and thus also inside a given invariant (ergodic) component, but changes from one component to another, also on irrational invariant tori. Thus if, specifically, $A_\omega(\mathbf{q}, \mathbf{p})$ is the characteristic function on a chaotic invariant component, so equal to $\chi_\omega(\mathbf{q}, \mathbf{p})$, then we have for the measure $\rho(\omega)$ of equation (8)

$$\begin{aligned}
\rho(\omega) &= \frac{\int_E d^N \mathbf{q} d^N \mathbf{p} \delta_1(E - H(\mathbf{q}, \mathbf{p})) \chi_\omega(\mathbf{q}, \mathbf{p})}{\int_E d^N \mathbf{q} d^N \mathbf{p} \delta_1(E - H(\mathbf{q}, \mathbf{p}))} \\
&= \frac{\int_{SOS} dX \tau \chi_\omega(\mathbf{q}, \mathbf{p})}{\int_{SOS} dX \tau} = \frac{\tau_\omega \int_{SOS} dX \chi_\omega(\mathbf{q}, \mathbf{p})}{\int_{SOS} dX \tau}
\end{aligned} \tag{51}$$

The third equality in the above equation follows by recalling that τ is constant on a given invariant component ω , equal to τ_ω . Of course, we can and should calculate (numerically) $\rho(\omega)$ by discretizing the SOS into a network of N_c cells of equal size $a = 1/N_c$. As is evident from equation (48), the invariant measure dX in a given cell can be chosen simply equal to

$$dX = gw(g)dg \tag{52}$$

This solves our problem of determining the classical measures $\rho(\omega)$ in the context of their role in the semiclassical limiting behaviour of the eigenstates and of the Hilbert space of a general, classically mixed, and therefore generic system, dealt with in the context of quantum chaos mainly in subsection 2.3.

5 Discussion and conclusions

The main purpose of this paper is first to provide a compact review of the main topics in the stationary quantum chaos in the general quantal Hamiltonian systems with discrete energy spectra, in correspondence with classical chaos, in the strict semiclassical limit where the effective Planck constant \hbar is sufficiently small. This problem has been expounded in the Introduction, section 1.

In section 2, we have explained the universality classes of spectral fluctuations and of their wave functions and of Wigner functions. These comprise of the classically integrable systems exhibiting Poisson statistics (subsection 2.1), and of the classically ergodic systems, obeying the statistics of the eigenvalues of the ensembles of random matrices from the classical Random Matrix Theories, namely GOE and GUE, depending on whether the system has or not an antiunitary symmetry, like e.g. the time reversal symmetry (subsection 2.2). Then we went on explaining the general case of classically mixed dynamics, in the transition region between classical integrability and ergodicity, very often well described by the scenario of the KAM Theory (subsection 2.3). We have shown how the Berry-Robnik (1984) picture can be applied, and have generalized it to arbitrary statistics, specifically $E(k, L)$ statistics. The main message is that the Hilbert space of eigenstates of a mixed system is

decomposed into the set of regular states (associated with a EBK/Maslov quantized invariant torus) and a set of irregular (chaotic) eigenstates, where their properties are captured by the most general semiclassical Wigner function in equation (5). The regular and irregular sequences are split exactly in proportion to the relative invariant Liouville measures of their supporting classical invariant sets. Then we have described some limitations to the universality classes and the semiclassical asymptotical behaviour (subsection 2.4). The first limitation stems from the existence of the Berry's outer energy scale (after unfolding) $L_{max} = (2\pi\hbar)/\langle\Delta E\rangle$ (see equation (11)), so that at $L \geq L_{max}$ we have no longer universality but saturation (e.g. of the sigma and of delta statistics). Nevertheless, as $\hbar \rightarrow 0$, L_{max} goes to infinity. The second limitation comes from the localization phenomena, controlled by the relation of the two time scales, the break time and the classical diffusion time. If break time for a given stationary eigenstate is shorter than diffusion (ergodic) time, then we have a strong localization. If the system is ergodic (but slowly diffusing), then we find a departure from the universal RMT statistics, and in extreme case can be close to Poisson statistics. In the opposite extreme, we find the extended states and correct behaviour generally described by the mixed case (**M**) of subsection 2.3. In subsection 2.5 we give some fundamental results on the statistical properties of general matrix elements, generalizing the important Feingold-Peres (1986) theory.

In section 3 we discuss the fundamental properties of the Wigner functions of the eigenstates, and introduce the *Principle of Uniform Semiclassical Condensation*. Then we say more about the wave functions and the spectral statistics, especially about the level spacing distributions, sigma and delta statistics. There we have also recalled Van Kampen's definition of quantum chaos and concluded that the quantum chaos, as a phenomenon "...causing the quantum systems to behave statistically...", does exist in the problem of the stationary Schrödinger equation and its solutions.

In section 4 we deal with the problem of how to determine, also numerically, the relative invariant measure of classically chaotic components, which enter in the semiclassical formulae for statistics of mixed systems (9). We have described the main ideas, the approach and the results of our recent works on this subject, giving the final description of how to proceed. The main goal of the theory of stationary quantum chaos is to explain and to describe the mixed systems. Then, the quantal parameters ρ_j determined by analyzing the spectral statistics must be equal to the purely classical relative invariant (Liouville) measures (subsection 2.3).

We believe that the material presented here is also a stimulation for further theoretical and numerical work, and also shows the need for more rigorous results on the side of the mathematical physics. For example, we still need the proof of the BGS-Conjecture, especially in its full generality presented in equation (4). We need

to prove the Principle of Uniform Semiclassical Condensation, which is more general than the BGS-Conjecture. Finally, as one of the most important issues, we need to understand more deeply and quantitatively the localization phenomena in the stationary eigenstates of general systems. Some important new steps in this direction have been recently undertaken by Casati and Prosen (1998a,b), and by Krylov and Robnik (1998). The most general aspects of quantum chaos related to other branches of theoretical and experimental physics have been recently reviewed and discussed in detail by Weidenmüller and coworkers (Guhr *et al* 1998).

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